CS2: A PIECEWISE-LINEAR MODEL FOR LARGE STRAIN CONSOLIDATION

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SUMMARY

This paper presents a piecewise-linear finite-difference model for one-dimensional large strain consolidation called CS2. CS2 is developed using a fixed Eulerian co-ordinate system and constitutive relationships which are defined by discrete data points. The model is dimensionless such that solutions are independent of the initial height of the compressible layer and the absolute magnitude of the hydraulic conductivity of the soil. The capability of CS2 is illustrated using four example problems involving small strain, large strain, self-weight, and non-linear constitutive relationships. In each case, the performance of the model is comparable to other available analytical and numerical solutions. Using CS2, correction factors are developed for the conventional Terzaghi theory which account for the effect of vertical strain on computed values of elapsed time and maximum excess pore pressure during consolidation. © 1997 by John Wiley & Sons, Ltd.

Int. J. Numer. Anal. Meth. Geomech., Vol. 21, 453–475 (1997)
(No. of Figures: 6   No. of Tables: 4   No. of Refs: 25)

Key words: consolidation; settlement; large strain; numerical modelling

1. INTRODUCTION

Although most developments concerning one-dimensional large strain consolidation have found the theory of Gibson *et al.*[1] a suitable point of departure,[2–14] Townsend and McVay[15] concluded that piecewise-linear models have greater versatility with regard to initial conditions, boundary conditions and soil heterogeneity than models based on material co-ordinates. In the piecewise-linear (or 'piecewise-iterative') approach, all variables pertaining to problem geometry, material properties, fluid flow and effective stress are updated after each time step with respect to a fixed Eulerian co-ordinate system.[16–18] Olson and Ladd[16] used the implicit Crank–Nicholson method to advance the solution forward in time, whereas Yong *et al.*[17] used the explicit finite-difference method. In either case, the time increment must be sufficiently small that all variables can be approximated as constant for each iteration. As a result, piecewise-linear models are generally considered to be computationally intensive in comparison to material co-ordinate-based models.[15] A second perceived limitation of the piecewise-linear method is that the cumulative effect of small incremental errors may invalidate the results of a numerical simulation.[19] However, Yong

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Contract grant sponsor: U.S. National Science Foundation; contract grant number: CMS 9309566; contract grant number: CMS 9622644

CCC 0363–9061/97/070453–23$17.50

Received 9 August 1996
Revised 18 November 1996

and Ludwig[18] showed that their piecewise-linear solution for the self-weight consolidation of an accumulating layer was equivalent to the solution given by the theory of Gibson *et al.*[6] The piecewise-linear method has also compared favourably to other large strain formulations in subsequent studies.[15,20]

This paper presents a dimensionless piecewise-linear numerical model for one-dimensional consolidation called Consolidation Settlement 2 (CS2). CS2 accounts for large strain, soil self-weight, the relative velocity of fluid and solid phases, and variable hydraulic conductivity and compressibility during the consolidation process. The constitutive relationships for the soil are specified using discrete data points, without requiring mathematical approximations or the calculation of derivative functions. CS2 is developed using a dimensionless approach by which solutions are independent of the initial height of the compressible layer and the absolute magnitude of the hydraulic conductivity of the soil. In this paper, the development of CS2 is first presented, followed by a discussion of the CS2 source code. Verification of the methodology is illustrated through a comparison of numerical solutions to four example problems involving small strain, large strain, self-weight and non-linear constitutive relationships. Lastly, correction factors are presented for the conventional Terzaghi theory which account for the effect of vertical strain of the compressible layer on the consolidation process.

2. MODEL DESCRIPTION

2.1. Geometry

The initial geometry of the compressible layer, prior to the application of a vertical stress increment (time $t < 0$), is shown in Figure 1(a). A saturated homogeneous soil stratum of infinite lateral extent and initial height H_o is treated as an idealized two-phase material in which the solid particles and the pore fluid are incompressible. The vertical Eulerian co-ordinate, z, is defined as

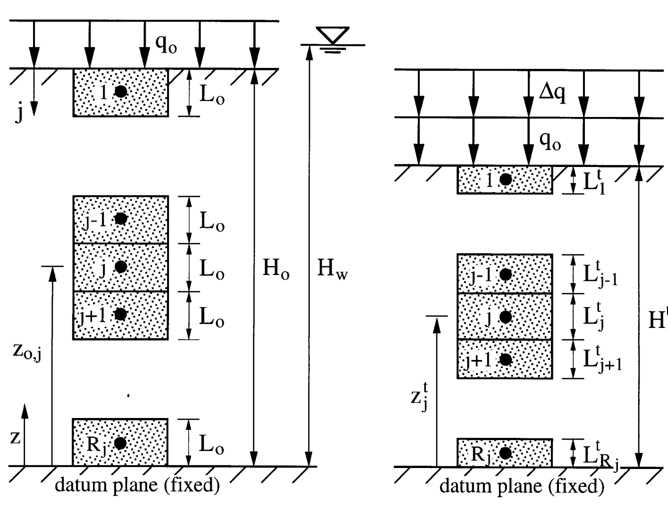

Figure 1. Geometry for CS2: (a) initial configuration ($t < 0$) and (b) configuration after application of vertical stress increment ($t \geqslant 0$)

positive upward (against gravity) from a fixed datum plane coincident with the bottom of the layer. The element co-ordinate, j, is directed downward from its origin at the top boundary. The stratum is vertically subdivided into R_j elements, each having unit cross-sectional area, constant initial height L_o, and a central node located at initial elevation $z_{o,j}$. The distribution of initial void ratio, $e_{o,j}$, within the layer is assumed to be in equilibrium with the effective overburden stress, q_o, and the self-weight of the soil. The top and bottom boundaries of the layer can be specified as drained or impermeable. Drainage boundaries are hydraulically connected to a groundwater table at constant elevation H_w. At $t = 0$, an instantaneous vertical effective stress increment, Δq, is applied to the compressible stratum. Thereafter, both q_o and Δq are constant and move with the upper boundary. At some later time t (Figure 1(b)), the height of the layer is H^t and the height of the jth element is L_j^t. Nodes translate vertically and remain at the centre of their respective elements throughout the consolidation process. Node elevations, z_j^t, are taken with respect to the fixed datum and are updated at each time step.

The following dimensionless quantities are defined for the initial configuration of the layer:

$$H_o^* = \frac{H_o}{H_o} = 1 \tag{1a}$$

$$H_w^* = \frac{H_w}{H_o} \tag{1b}$$

$$L_o^* = \frac{L_o}{H_o} \tag{1c}$$

$$z_{o,j}^* = \frac{z_{o,j}}{H_o}, \quad j = 1, 2, \ldots, R_j \tag{1d}$$

$$q_o^* = \frac{q_o}{H_o \gamma_w} \tag{1e}$$

where γ_w is the unit weight of water (constant). Thus, in CS2, the compressible layer has an initial height of unity, and the corresponding dimensionless lengths and stresses are scaled accordingly. A dimensionless time, τ, is defined as

$$\tau = \frac{k_{q_o} t}{H_o} \tag{2}$$

where k_{q_o} is the vertical hydraulic conductivity at e_{q_o}, and e_{q_o} is the void ratio at vertical effective stress q_o. For $\tau \geqslant 0$, the following dimensionless quantities are defined:

$$H^{*\tau} = \frac{H^\tau}{H_o} \tag{3a}$$

$$L_j^{*\tau} = \frac{L_j^\tau}{H_o}, \quad j = 1, 2, \ldots, R_j \tag{3b}$$

$$z_j^{*\tau} = \frac{z_j^\tau}{H_o}, \quad j = 1, 2, \ldots, R_j \tag{3c}$$

$$\Delta q^* = \frac{\Delta q}{H_o \gamma_w} \tag{3d}$$

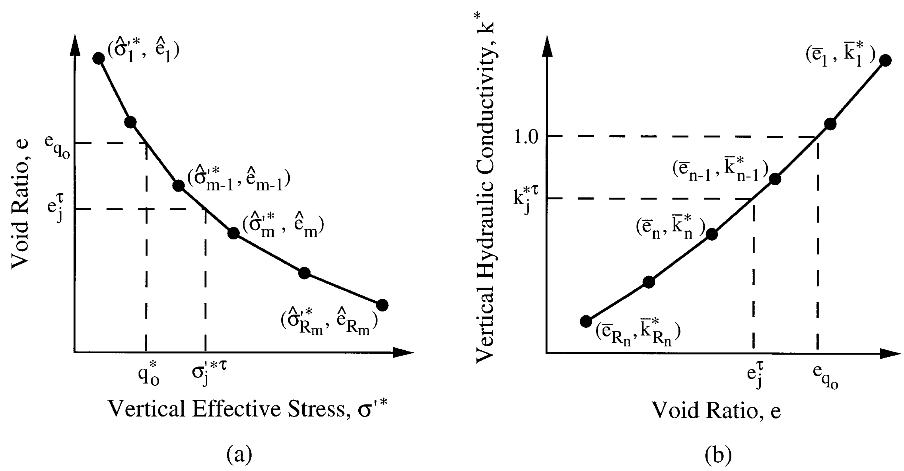

Figure 2. Soil constitutive relationships: (a) compressibility and (b) hydraulic conductivity

2.2. Constitutive relationships

The constitutive relationships for the compressible layer are shown in Figure 2. The compressibility curve (Figure 2(a)) is defined by R_m ($\geqslant 2$) pairs of corresponding void ratio, $\hat{e}$, and dimensionless vertical effective stress, $\hat{\sigma}'^*$, where

$$\hat{\sigma}'^* = \frac{\hat{\sigma}'}{H_o \gamma_w} \tag{4}$$

The hydraulic conductivity relationship (Figure 2(b)) is defined by R_n ($\geqslant 2$) pairs of corresponding void ratio, $\bar{e}$, and dimensionless vertical hydraulic conductivity, $\bar{k}^*$, where

$$\bar{k}^* = \frac{\bar{k}}{k_{q_o}} \tag{5}$$

The superscripts ^ and ¯ are used to distinguish the input values of void ratio, vertical effective stress and vertical hydraulic conductivity that define these curves. A one-to-one correspondence is assumed for each constitutive relationship shown in Figure 2. Thus, CS2 does not account for the effects of strain rate, secondary compression or aging on the compressibility or hydraulic conductivity of the soil. The relationships can, however, take nearly any desired form by choosing an appropriate number of data points. The only restriction is that the ordinate of each curve must vary monotonically with the corresponding abscissa (i.e., $\hat{e}$ must continuously decrease with increasing $\hat{\sigma}'^*$, and $\bar{k}^*$ must continuously decrease with decreasing $\bar{e}$).

2.3. Total stress, effective stress, and pore pressure

The vertical total stress at each node is computed from the applied overburden stress and the self-weight of the compressible layer. For $\tau \geqslant 0$, the dimensionless total stress at node j, $\sigma_j^{*\tau}$, is

$$\sigma_j^{*\tau} = \frac{\sigma_j^\tau}{H_o \gamma_w} = H_w^* - H^{*\tau} + q_o^* + \Delta q^* + \frac{L_j^{*\tau} \gamma_j^{*\tau}}{2} + \sum_{i=1}^{j-1} L_i^{*\tau} \gamma_i^{*\tau}, \quad j = 1, 2, \ldots, R_j \tag{6}$$

where $\gamma_j^{*\tau}$ is the dimensionless saturated unit weight of element j,

$$\gamma_j^{*\tau} = \frac{\gamma_j^\tau}{\gamma_w} = \frac{G_s + e_j^\tau}{1 + e_j^\tau}, \quad j = 1, 2, \ldots, R_j \tag{7}$$

and e_j^τ is the corresponding void ratio. In CS2, the specific gravity of soil solids, G_s, is constant for the compressible layer, and e_j^τ is constant within each element over any given time increment.

The dimensionless vertical effective stress at node j, $\sigma_j'^{*\tau}$, is computed from e_j^τ and the compressibility curve as

$$\sigma_j'^{*\tau} = \frac{\sigma_j'^\tau}{H_o \gamma_w} = \hat{\sigma}_{m-1}'^{*} + \frac{\hat{e}_{m-1} - e_j^\tau}{a_{v,j}^{*\tau}}, \quad j = 1, 2, \ldots, R_j \tag{8}$$

where the dimensionless coefficient of compressibility, $a_{v,j}^{*\tau}$, is

$$a_{v,j}^{*\tau} = -\frac{\hat{e}_m - \hat{e}_{m-1}}{\hat{\sigma}_m'^{*} - \hat{\sigma}_{m-1}'^{*}}, \quad j = 1, 2, \ldots, R_j \tag{9}$$

and the points $(\hat{\sigma}_{m-1}'^{*}, \hat{e}_{m-1})$ and $(\hat{\sigma}_m'^{*}, \hat{e}_m)$ define the linear segment of the compressibility curve such that $\hat{e}_m \leqslant e_j^\tau \leqslant \hat{e}_{m-1}$ (Figure 2(a)).

The dimensionless pore pressure for node j, $u_j^{*\tau}$, is the difference between the total and effective stress,

$$u_j^{*\tau} = \frac{u_j^\tau}{H_o \gamma_w} = \sigma_j^{*\tau} - \sigma_j'^{*\tau}, \quad j = 1, 2, \ldots, R_j \tag{10}$$

and the corresponding dimensionless excess pore pressure is

$$u_{ex,j}^{*\tau} = \frac{u_{ex,j}^\tau}{H_o \gamma_w} = u_j^{*\tau} + z_j^{*\tau} - H_w^{*}, \quad j = 1, 2, \ldots, R_j \tag{11}$$

CS2 calculates $u_{ex,j}^{*\tau}$ to determine the local degree of consolidation in terms of per cent pore pressure dissipation (equation (27b)). The solution method is otherwise based on the 'total' pore pressure $u_j^{*\tau}$. This approach is consistent with the recent recommendation of Schiffman *et al.*,[14] who favour abandoning the use of excess pore pressure when analysing field problems and working exclusively in terms of total pore pressure.

2.4. Hydraulic conductivity

Similar to equation (8), the dimensionless vertical hydraulic conductivity for element j, $k_j^{*\tau}$, is calculated as

$$k_j^{*\tau} = \frac{k_j^\tau}{k_{q_o}} = \bar{k}_{n-1}^{*} + \frac{\bar{k}_n^{*} - \bar{k}_{n-1}^{*}}{\bar{e}_n - \bar{e}_{n-1}}(e_j^\tau - \bar{e}_{n-1}), \quad j = 1, 2, \ldots, R_j \tag{12}$$

where the points $(\bar{e}_{n-1}, \bar{k}_{n-1}^{*})$ and $(\bar{e}_n, \bar{k}_n^{*})$ define the linear segment of the hydraulic conductivity curve such that $\bar{e}_n \leqslant e_j^\tau \leqslant \bar{e}_{n-1}$ (Figure 2(b)). As the hydraulic conductivity of contiguous elements will generally not be equal, an equivalent series hydraulic conductivity is used to compute

inter-element fluid flow. At time τ, the equivalent dimensionless series hydraulic conductivity, $k^{*\tau}_{s,j}$, between node j and the immediately overlying node $(j-1)$ is

$$k^{*\tau}_{s,j} = \frac{k^{*\tau}_{j-1} k^{*\tau}_{j} (L^{*\tau}_{j-1} + L^{*\tau}_{j})}{L^{*\tau}_{j-1} k^{*\tau}_{j} + L^{*\tau}_{j} k^{*\tau}_{j-1}}, \quad j = 2, 3, \ldots, R_j. \tag{13}$$

2.5. Mass and momentum conservation

For CS2, the weight of solids contained within each element is invariant throughout the consolidation process (see equation (25)). Thus, solid particles do not cross from one element to the next, and the element interfaces, as well as the nodes, can be considered to be embedded within the soil skeleton. As such, consideration of the relative fluid velocity across the upper and lower boundaries of each element is sufficient for the mass balance computation.

Neglecting inertial forces, fluid flow within a porous medium in which the solid phase is also in motion is governed by the Darcy–Gersevanov law[21,22]

$$n(\tilde{v}_f - \tilde{v}_s) = -ki \tag{14}$$

where n is the porosity, $\tilde{v}_f$ is the seepage velocity of the fluid (based on the average void area per unit area of soil), $\tilde{v}_s$ is the velocity of solids, k is the hydraulic conductivity, and i is the hydraulic gradient, Both $\tilde{v}_f$ and $\tilde{v}_s$ are absolute velocities with respect to the fixed datum at the base of the stratum. Recalling the principle of relative motion, $\tilde{v}_f$ can be written as

$$\tilde{v}_f = \tilde{v}_{rf} + \tilde{v}_s = \frac{v_{rf}}{n} + \tilde{v}_s \tag{15}$$

where $\tilde{v}_{rf}$ and v_{rf} are, respectively, the relative seepage velocity and the relative discharge velocity of the pore fluid with respect to the solid phase. Substituting equation (15) into equation (14), the relative discharge velocity is

$$v_{rf} = -ki \tag{16}$$

In CS2, the dimensionless relative discharge velocity (positive upward), $v^{*\tau}_{rf,j}$, between nodes j and $j-1$ (Figure 3) is

$$v^{*\tau}_{rf,j} = \frac{v^{\tau}_{rf,j}}{k_{q_o}} = -k^{*\tau}_{s,j} i^{\tau}_{j}, \quad j = 2, 3, \ldots, R_j \tag{17}$$

where the hydraulic gradient, i^{τ}_{j}, is

$$i^{\tau}_{j} = \frac{h^{*\tau}_{j-1} - h^{*\tau}_{j}}{z^{*\tau}_{j-1} - z^{*\tau}_{j}}, \quad j = 2, 3, \ldots, R_j \tag{18}$$

and the dimensionless total head for node j, $h^{*\tau}_{j}$, is

$$h^{*\tau}_{j} = \frac{h^{\tau}_{j}}{H_o} = z^{*\tau}_{j} + u^{*\tau}_{j}, \quad j = 1, 2, \ldots, R_j \tag{19}$$

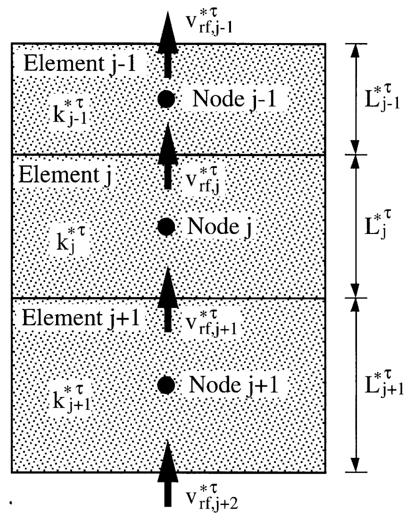

Figure 3. Fluid flow between contiguous elements

The following additional equations are required at the upper and lower boundaries of the stratum:

$$i_1^\tau = \frac{H_w^* - h_1^{*\tau}}{H^{*\tau} - z_1^{*\tau}} \quad \text{hydraulic gradient at top boundary (drained)} \tag{20a}$$

$$i_1^\tau = 0 \quad \text{hydraulic gradient at top boundary (undrained)} \tag{20b}$$

$$i_b^\tau = \frac{h_{R_j}^{*\tau} - H_w^*}{z_{R_j}^{*\tau}} \quad \text{hydraulic gradient at bottom boundary (drained)} \tag{20c}$$

$$i_b^\tau = 0 \quad \text{hydraulic gradient at bottom boundary (undrained)} \tag{20d}$$

$$v_{rf,1}^{*\tau} = -\,k_1^{*\tau} i_1^\tau \quad \text{dimensionless relative discharge velocity at top boundary} \tag{20e}$$

$$v_{rf,b}^{*\tau} = -\,k_{R_j}^{*\tau} i_b^\tau \quad \text{dimensionless relative discharge velocity at bottom boundary} \tag{20f}$$

Using the method of explicit direct integration, the dimensionless height of element j at time $\tau + \Delta\tau$ is

$$L_j^{*\tau+\Delta\tau} = L_j^{*\tau} - (v_{rf,j}^{*\tau} - v_{rf,j+1}^{*\tau})\Delta\tau, \quad j = 1, 2, \ldots, R_j - 1 \tag{21}$$

where $\Delta\tau$ is the dimensionless time increment. At the bottom of the stratum, the corresponding equation for element R_j is

$$L_{R_j}^{*\tau+\Delta\tau} = L_{R_j}^{*\tau} - (v_{rf,R_j}^{*\tau} - v_{rf,b}^{*\tau})\Delta\tau \tag{22}$$

The new void ratio for each element j at time $\tau + \Delta\tau$ is

$$e_j^{\tau+\Delta\tau} = \frac{L_j^{*\tau+\Delta\tau}(1 + e_{o,j})}{L_o^*} - 1, \quad j = 1, 2, \ldots, R_j \tag{23}$$

and the corresponding dimensionless weight of solids (per unit area), $W^{*\tau+\Delta\tau}_{s,j}$, is

$$W^{*\tau+\Delta\tau}_{s,j} = \frac{W^{\tau+\Delta\tau}_{s,j}}{H_o \gamma_w} = \frac{G_s}{1 + e^{\tau+\Delta\tau}_j} L^{*\tau+\Delta\tau}_j, \quad j = 1, 2, \ldots, R_j \tag{24}$$

Combining equations (23) and (24),

$$W^{*\tau+\Delta\tau}_{s,j} = \frac{G_s}{1 + e_{o,j}} L^*_o, \quad j = 1, 2, \ldots, R_j \tag{25}$$

which is equivalent to the initial dimensionless weight of solids for element j. Thus, the solid mass is invariant for each element throughout the consolidation process.

2.6. Settlement

At each time step, the dimensionless height of the compressible layer, the dimensionless settlement and the average degree of consolidation are, respectively,

$$H^{*\tau} = \sum_{j=1}^{R_j} L^{*\tau}_j \tag{26a}$$

$$S^{*\tau} = \frac{S^\tau}{H_o} = 1 - H^{*\tau} \tag{26b}$$

$$U^\tau_{avg} = \frac{S^{*\tau}}{S^*} \tag{26c}$$

where S^τ is the settlement at time τ, S^* is the final strain of the compressible layer ($= S/H_o$) and S is the total settlement at the end of consolidation. The local degree of consolidation is defined with respect to void ratio, $U^\tau_{e,j}$, and excess pore pressure, $U^\tau_{u,j}$, as

$$U^\tau_{e,j} = \frac{e_{o,j} - e^\tau_j}{e_{o,j} - e_{f,j}}, \quad j = 1, 2, \ldots, R_j \tag{27a}$$

$$U^\tau_{u,j} = 1 - \frac{u^{*\tau}_{ex,j}}{\Delta q^*}, \quad j = 1, 2, \ldots, R_j \tag{27b}$$

where $e_{f,j}$ is the final void ratio for element j at the end of consolidation.

2.7. Time increment

At each time step, CS2 uses two criteria to calculate the dimensionless time increment, $\Delta\tau$. The first criterion is adapted from the following requirement for a numerically stable solution of the parabolic Terzaghi consolidation equation using the explicit finite-difference method:[23]

$$\Delta t = \frac{\alpha(\Delta z)^2}{c_v} \tag{28}$$

where α is a constant $\leqslant 0{\cdot}5$, $c_{\rm v}$ is the coefficient of consolidation and Δz is the vertical node spacing. For CS2, a dimensionless coefficient of consolidation for element j, $c^{*\tau}_{{\rm v},j}$, is defined as

$$c^{*\tau}_{{\rm v},j} = \frac{c^{\tau}_{{\rm v},j}}{k_{q_{\rm o}} H_{\rm o}} = \frac{k^{*\tau}_j (1 + e^{\tau}_j)}{a^{*\tau}_{{\rm v},j}}, \quad j = 1, 2, \ldots, R_j \tag{29}$$

Combining equations (28) and (29), and approximating Δz as $L^{*\tau}_j H_{\rm o}$, the dimensionless time increment for element j, $\Delta\tau_j$, is

$$\Delta\tau_j = \frac{k_{q_{\rm o}} \Delta t}{H_{\rm o}} = \frac{\alpha a^{*\tau}_{{\rm v},j} (L^{*\tau}_j)^2}{k^{*\tau}_j (1 + e^{\tau}_j)}, \quad j = 1, 2, \ldots, R_j \tag{30}$$

Numerical studies[24] have shown that CS2 is stable for $\alpha < 0{\cdot}5$, and is most accurate for $\alpha \approx 0{\cdot}4$. However, depending on the value of Δq^*, time increments calculated using equation (30) may be too large to resolve properly the high initial discharge velocity near the drainage boundaries. Thus, a second time increment criterion is defined to increase the accuracy of the simulation during the early stages of consolidation:

$$\Delta\tau_j = \left| \frac{0{\cdot}01 L^*_{\rm o} (e_{{\rm o},j} - e_{{\rm f},j})}{(1 + e_{{\rm o},j})(v^{*\tau}_{{\rm rf},j} - v^{*\tau}_{{\rm rf},j+1})} \right|, \quad j = 1, 2, \ldots, R_j - 1 \tag{31a}$$

$$\Delta\tau_{R_j} = \left| \frac{0{\cdot}01 L^*_{\rm o} (e_{{\rm o},R_j} - e_{{\rm f},R_j}}{(1 + e_{{\rm o},R_j})(v^{*\tau}_{{\rm rf},R_j} - v^{*\tau}_{{\rm rf,b}})} \right| \tag{31b}$$

where the vertical bars signify absolute value. Equation (31) ensures that, over each time increment, the change in vertical strain for each element will not exceed 1 per cent of the corresponding final vertical strain for that same element. Using equations (30) and (31), CS2 performs a search at each time step to find the smallest value of $\Delta\tau_j$, which is then used to advance the solution forward in time for all elements.

3. CS2 COMPUTER PROGRAM

The CS2 source code, less the output statements and procedures, is listed in Appendix II. A flow chart illustrating the basic algorithm for the program is shown in Figure 4. The required input data are the number of elements (R_j), the applied stress conditions ($q^*_{\rm o}, \Delta q^*$), the specific gravity of solids ($G_{\rm s}$), the elevation of the groundwater table ($H^*_{\rm w}$), the data points for the constitutive relationships, the boundary drainage conditions and the termination criteria for the program. As CS2 is cast in dimensionless form, the solution is independent of $H_{\rm o}$ and $k_{q_{\rm o}}$. The choice for R_j depends on the desired solution accuracy and the acceptable computation time. Generally, a value of R_j between 50 and 100 has been found to give satisfactory results.[24] After CS2 reads the input data, an iterative procedure is used to compute a distribution of $e_{{\rm o},j}$ that is consistent with the specified compressibility curve, the self-weight of the soil and $q^*_{\rm o}$. Once the values of $e_{{\rm o},j}$ are known, $e_{{\rm f},j}$ and S^* are calculated for the stress increment Δq^*. To begin the main calculation loop, the elevation and total stress are computed for each node. The effective stress, hydraulic conductivity and coefficient of compressibility are then calculated for each element from the corresponding initial void ratio and the constitutive relationships. The distribution of total head is used to compute flow between contiguous elements, and the vertical compression of each element is calculated from the net fluid outflow during time increment $\Delta\tau$. New element heights

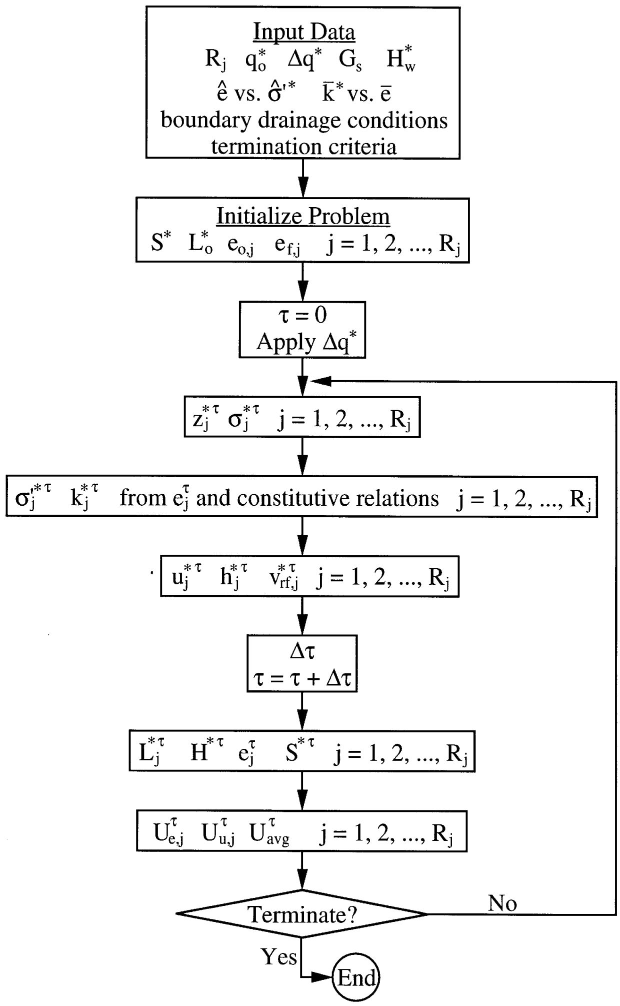

Figure 4. Flow chart for CS2

and void ratios are calculated, as well as the settlement and the local and average degrees of consolidation. Program execution terminates when $\tau \geqslant \tau_{final}$ or $U^{\tau}_{avg} \geqslant U_{avg-final}$, where τ_{final} and $U_{avg-final}$ are user-specified values. If neither condition is satisfied, CS2 repeats the calculation sequence with new values of e_j, L_j^* and H^*.

4. MODEL VERIFICATION

The performance of CS2 is illustrated using four example problems. Table I gives the input data for each problem. Additional verification checks of CS2 and comparisons with other available solutions are presented by Berles.[24]

Table I. CS2 example problems

Variable	Problem 1 (small strain) c_v = constant	Problem 2 (large strain) c_v = constant	Problem 3 (large strain) c_F = constant	Problem 4 (large strain with self-weight)
R_j	20, 50, 100, 200	50, 100	50, 100	100
e_{q_o}	1·0	1·0	1·0	18·8
q_o^*	1·020	1·020	1·020	0·00361
Δq^*	$5{\cdot}099 \times 10^{-6}$	4·079	1·020, 4·079	0
G_s	1·0	1·0	1·0	2·71
H_w^*	1·0	1·0	1·0	1·0
Compressibility	Linear $a_v^* = 0{\cdot}1961$	Linear $a_v^* = 0{\cdot}1961$	Linear $a_v^* = 0{\cdot}1961$	Non-linear Equation (36a)
Hydraulic conductivity	Constant $k_{q_o} = 1 \times 10^{-9}$ m/s $k^* = 1{\cdot}0$	Variable $k_{q_o} = 1 \times 10^{-9}$ m/s Equation (33)	Variable $k_{q_o} = 1 \times 10^{-9}$ m/s Equation (35)	Variable $k_{q_o} = 2{\cdot}43 \times 10^{-6}$ m/s Equation (36b)
c_v^*	10·197	10·197	Variable	Variable
c_F^*	10·197	Variable	10·197	Variable
Boundary conditions	Double-drained	Single-drained at either boundary	Single-drained at either boundary	Single-drained at top boundary
S^*	0·00005%	40%	10%, 40%	46%

4.1. Small strains

Problem 1 involves small strains and is well suited to the classical Terzaghi theory of consolidation. A double-drained compressible layer has an initial height of 20 m and is in equilibrium under an effective overburden stress of 200 kPa The elevation of the groundwater table is 20 m, the self-weight of the soil is neglected ($G_s = 1$) and the initial void ratio is equal to 1·0 over the entire depth of the layer. At $t = 0$, a uniform vertical effective stress increment of 0·001 kPa is applied to the stratum. Throughout the subsequent consolidation process, the coefficient of compressibility and the hydraulic conductivity are both constant, having values of 0·001/kPa and 1×10^{-9} m/s, respectively. Once consolidation is completed, the final strain, S^*, is 0·00005 per cent, where

$$S^* = \frac{a_v^* \Delta q^*}{1 + e_{q_o}} \tag{32}$$

Five solutions to problem 1 are shown in Table II. For each, U_{avg} is listed as a function of the dimensionless Terzaghi time factor for a double-drained layer ($4c_v^* \tau$). The second column gives the analytical solution, and the remaining columns show the numerical values computed using CS2 with 20, 50, 100 and 200 elements. Each solution is in good agreement with the exact value, and the accuracy improves as the number of elements increase. Using a 120 MHz desktop computer, the required computation time for these solutions was 1, 3, 14 and 107 s for $R_j = 20, 50, 100$ and 200, respectively. Thus, the disadvantage of long computation time generally associated with piecewise-linear models[15] may be less important than in the past due to the increasing capability of desktop computers.

Table II. Comparison of solutions for problem 1 (small strain)

Time factor $4c_v^* \tau$	Average degree of consolidation, U_{avg} (%)				
	Terzaghi analytical solution	CS2 $R_j = 20$	CS2 $R_j = 50$	CS2 $R_j = 100$	CS2 $R_j = 200$
0·005	7·979	6·765	7·836	7·958	7·976
0·01	11·284	10·515	11·199	11·274	11·282
0·05	25·231	24·977	25·215	25·228	25·231
0·1	35·682	35·550	35·673	35·680	35·682
0·15	43·695	43·612	43·688	43·694	43·695
0·2	50·409	50·352	50·404	50·408	50·409
0·3	61·324	61·304	61·324	61·324	61·324
0·5	76·395	76·416	76·400	76·396	76·395
0·8	88·740	88·774	88·747	88·742	87·741
1·2	95·803	95·828	95·808	95·805	95·804

4.2. Large strains

The initial condition for problems 2 and 3, involving large strain and no self-weight, is identical to that for problem 1. Final strains of 10 and 40 per cent are achieved by applying stress increments of 200 and 800 kPa, respectively, to the compressible stratum. CS2 solutions are compared to solutions published by Lee and Sills[25] for the special cases of constant c_v and constant c_F. For problem 2, $c_{v,j}^{*\tau}$ is held constant by adjusting $k_j^{*\tau}$ at each time step according to the following equation:

$$k_j^{*\tau} = k_{q_o}^* \frac{1 + e_{q_o}}{1 + e_j^\tau}, \quad j = 1, 2, \ldots, R_j \tag{33}$$

where $k_{q_o}^* = 1{\cdot}0$. Five numerical solutions for problem 2 are compared in Table III. Column 2 lists values of U_{avg} obtained from an approximate large strain solution which is valid for the early stages of consolidation. Corresponding values of U_{avg} obtained using the theory of Gibson *et al.*[1] and the moving boundary theory of Lee and Sills[25] are shown in Columns 3 and 4, respectively. The last two columns give the CS2 solutions for $R_j = 50$ and 100. Table III shows that the CS2 values are in good agreement with the other available solutions for this problem. Similar to Table II, the CS2 solutions improve slightly as R_j increases.

Problem 3 represents the strain-invariant case for the theory of Gibson *et al.*[1] corresponding to constant c_F, were $c_{F,j}^{*\tau}$ of element j is defined as

$$c_{F,j}^{*\tau} = \frac{k_j^{*\tau}(1 + e_{q_o})^2}{a_{v,j}^{*\tau}(1 + e_j^\tau)}, \quad j = 1, 2, \ldots, R_j \tag{34}$$

To maintain constant $c_{F,j}^{*\tau}$, $k_j^{*\tau}$ is adjusted at each time step according to the following equation:

$$k_j^{*\tau} = k_{q_o}^* \frac{1 + e_j^\tau}{1 + e_{q_o}}, \quad j = 1, 2, \ldots, R_j \tag{35}$$

Table IV compares numerical solutions given by CS2 and the theories of Gibson *et al.*[1] and Lee and Sills.[25] The Gibson *et al.* solution holds for any S^*. The Lee and Sills and CS2 solutions are

Table III. Comparison of solutions for problem 2 (constant c_v)

Time factor $c_v^* \tau$	Average degree of consolidation, U_{avg} (%)				
	Small time approximation†	Gibson *et al.*[1] theory†	Lee and Sills[25] theory†	CS2 $R_j = 50$	CS2 $R_j = 100$
0·005	10·965	11·040	11·107	10·829	10·926
0·01	15·506	15·549	15·619	15·404	15·477
0·02	21·929	21·952	22·016	21·851	21·907
0·04	31·013	31·024	31·078	30·953	30·996
0·07	41·026	41·031	41·077	40·979	41·013
0·1	49·035	49·035	49·077	48·993	49·022
0·15	60·056	59·992	60·029	59·960	59·984
0·2	69·349	68·944	68·978	68·921	68·942
0·3		82·122	82·149	82·113	82·127
0·5		94·847	94·858	94·848	94·852
0·9		99·651	99·653	99·652	99·652

† Numerical values from Table 1 of Lee and Sills[25]

Table IV. Comparison of solutions for problem 3 (constant c_F)

Time factor $c_F^* \tau$	Average degree of consolidation, U_{avg} (%)					
	Gibson *et al.*[1] theory All S^*	Lee and Sills[25] theory† $S^* = 10\%$	Lee and Sills[25] theory† $S^* = 40\%$	CS2 $R_j = 50$ $S^* = 10\%$	CS2 $R_j = 50$ $S^* = 40\%$	CS2 $R_j = 100$ $S^* = 40\%$
0·01	11·284	11·315	11·292	11·274	11·274	11·282
0·03	19·544	19·558	19·488	19·540	19·540	19·543
0·06	27·639	27·645	27·533	27·637	27·637	27·639
0·1	35·682	35·683	35·538	35·680	35·680	35·682
0·15	43·695	43·693	43·522	43·694	43·694	43·695
0·25	56·223	56·216	55·976	56·223	56·223	56·223
0·41	70·525	70·333	70·073	70·526	70·526	70·525
0·61	82·006	81·886	81·682	82·008	82·008	82·006
0·81	89·015	88·941	88·805	89·016	89·016	89·015
1·01	93·293	93·248	93·162	93·295	93·295	93·294
1·41	97·500	97·483	97·451	97·501	97·501	97·501
2·01	99·431	99·427	99·420	99·432	99·432	99·431

† Numerical values from Table 2 of Lee and Sills[25]

shown for $S = 10$ and 40 per cent. The numerical values from all three theories are in good agreement. Table IV shows that the Lee and Sills solution is nearly invariant with strain, whereas the CS2 solution is precisely invariant (for constant R_j).

4.3. *Self-weight*

The data for problem 4 are taken from a self-weight consolidation field tank test described by McVay *et al.*[20] A single-drained column of phosphatic clay has an initial height of 6·33 m and an

initial uniform void ratio equal to 18·8. No stress is applied to the top of the column, and the clay consolidates under its own self-weight. The compressibility and hydraulic conductivity constitutive relationships for the clay are

$$e = 12{\cdot}19\ (\sigma'(\mathrm{kPa}))^{-0{\cdot}29} \tag{36a}$$

$$k\ (\mathrm{m/s}) = 1{\cdot}41 \times 10^{-11} e^{4{\cdot}11} \tag{36b}$$

For CS2, equations (36a) and (36b) were approximated using 42 data points each. Equation (36a) is inconsistent with the initial void ratio of the clay and the zero effective stress condition at the top of the column. Consequently, using equation (36a) and $e = 18{\cdot}8$, $q_o = 0{\cdot}224$ kPa was specified for problem 4. A small modification to the CS2 source code (not shown in Appendix II) was necessary to accommodate the uniform initial void ratio for this problem. Figure 5(a) shows

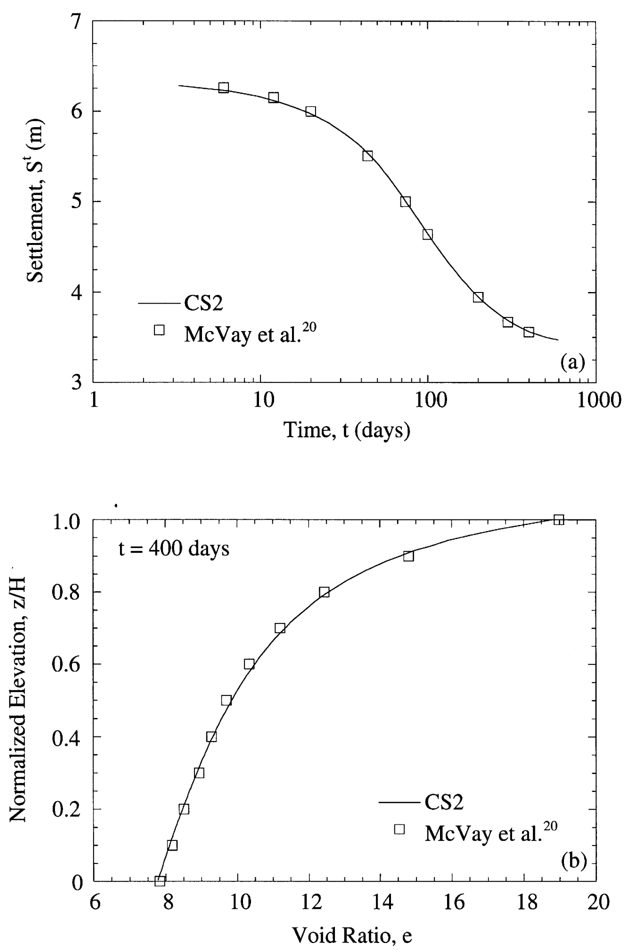

Figure 5. Problem 4: (a) settlement vs. time and (b) void ratio distribution at 400 days (individual data points scaled from Figures 11 and 12 of McVay *et al.*[20])

a comparison of the settlement vs. time curve obtained using CS2 with the numerical solutions presented by McVay *et al.*[20] Figure 5(b) compares the computed distributions of void ratio at 400 days elapsed time. The values are in close agreement for each plot.

A comparison of numerical and analytical solutions for problems 1–4 demonstrates that CS2 provides accurate estimates of (i) total settlement, (ii) rate of settlement and (iii) the distribution of spatially dependent variables, such as void ratio, as a function of time. It is therefore concluded that CS2 represents a correct formulation for large strain one-dimensional consolidation including the effects of soil self-weight.

5. PURE STRAIN EFFECT

In the strict sense, conventional consolidation theory is a zero-strain formulation because the length of the maximum drainage path is assumed to be constant throughout the consolidation process. Finite strain models, on the other hand, account for the effect of decreasing layer height during consolidation. In the remainder of this paper, CS2 is used to illustrate the effect of vertical strain of the compressible layer on the consolidation process — the so-called 'pure strain effect'.[25]

To investigate the pure strain effect, Berles[24] studied a series of consolidation problems which satisfy the assumptions of the conventional Terzaghi theory (i.e. constant a_v, constant k and no self-weight), with the exception that the final strain of the compressible layer is variable. For each problem, conventional theory was used to compute U_{avg} and the maximum excess pore pressure as a function of time for the case of infinitesimal strain. Using CS2, corresponding solutions were obtained for variable Δq^* such that S^* ranged from zero to the maximum possible value $(e_{q_o}/(1 + e_{q_o}))$. The ratio of elapsed time for finite strain (CS2) and infinitesimal strain (Terzaghi) solutions at corresponding values of U_{avg} is defined as t^r. For example, the ratio of elapsed time at $U_{avg} = 90$ per cent, t^r_{90}, is

$$t^r_{90} = \frac{t_{90}(\text{CS2})}{t_{90}(\text{Terzaghi})} \tag{37}$$

Berles[24] found that, for any given U_{avg} and S^*, t^r is independent of effective stress conditions, material properties and boundary drainage conditions. Figure 6(a) shows the relationship for t^r as a function of U_{avg} and S^*. For the case of infinitesimal strain ($S^* = 0$), the Terzaghi theory and CS2 give identical results, and $t^r = 1{\cdot}0$ for all U_{avg}. As S^* increases, the drainage path length decreases during consolidation, which consequently increases the rate of settlement for CS2 and decreases t^r. Interestingly, the curves for $U_{avg} = 30$ and $U_{avg} = 50$ per cent are nearly indistinguishable, illustrating that the relationship between t^r and U_{avg} is non-linear for any given S^*. The curves shown in Figure 6(a) are not strongly a function of U_{avg}, and can be characterized, as a first approximation, by the following equation:

$$t^r = 1 - 0{\cdot}75S^* \tag{38}$$

The effect of layer strain on the calculated value of maximum excess pore pressure, $u_{ex,\,max}$, is shown in Figure 6(b). In this case, a maximum excess pore pressure ratio, u^r_{ex}, is defined as

$$u^r_{ex} = \frac{u_{ex,\,max}/\Delta q \ \ (\text{CS2})}{u_{ex,\,max}/\Delta q \ \ (\text{Terzaghi})} \tag{39}$$

For double-drained problems, $u_{ex,\,max}$ occurs at the mid-height of the layer, whereas $u_{ex,\,max}$ is located at the impermeable boundary for the single-drained case. For $S^* = 0$, the Terzaghi theory

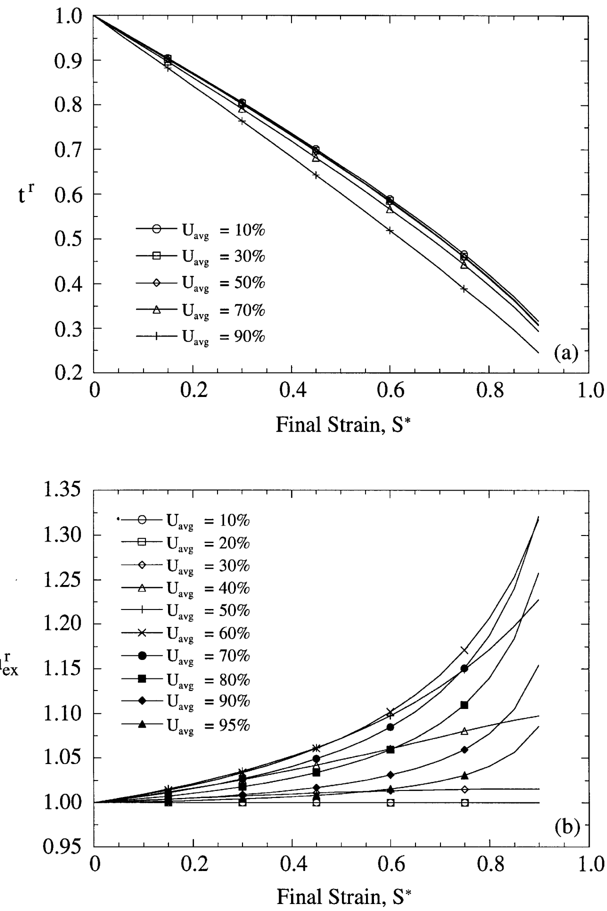

Figure 6. Correction factors for pure strain effect: (a) time and (b) maximum excess pore pressure

and CS2 again give identical results, and $u^{r}_{ex} = 1{\cdot}0$ for all U_{avg}. As S^* increases, u^{r}_{ex} increases to values larger than 1·0. Thus, for any given U_{avg}, the maximum excess pore pressure for the finite strain case is larger than the corresponding maximum excess pore pressure predicted using conventional theory. This results from the significantly shorter time needed to reach any given U_{avg} when vertical strain is taken into account (Figure 6(a)). Figure 6(b) shows the error for conventional theory is largest for U_{avg} values of approximately 60 per cent. However, for $S^* < 0{\cdot}4$, the maximum error is less than 5 per cent for all U_{avg}.

The t^{r} and u^{r}_{ex} factors shown in Figure 6 may be used to correct conventional theory estimates of settlement and maximum excess pore pressure for the effect of vertical strain of the compressible layer. In essence, this analysis has characterized the effect of one additional degree of freedom, S^*, for the Terzaghi consolidation theory. For actual finite strain consolidation problems, significant changes in compressibility and hydraulic conductivity may occur during the consolidation process that are not taken into account by the correction factors shown in Figure 6.

6. CONCLUSIONS

The following conclusions are reached as a result of this investigation of one-dimensional large strain consolidation:

(1) CS2 is a dimensionless piecewise-linear finite-difference model which accounts for large strain, self-weight, the relative velocity of fluid and solid phases, and variable hydraulic conductivity and compressibility during the consolidation process. A conventional Eulerian co-ordinate system is adopted for CS2 in which node elevations are taken with respect to a fixed datum, and all variables refer to the current configuration of the system. The constitutive relationships for the soil are specified using discrete data points, without requiring mathematical approximations or the calculation of derivative functions. CS2 is cast in dimensionless form such that solutions are independent of the initial height of the compressible layer and the absolute magnitude of the hydraulic conductivity of the soil.
(2) The performance of CS2 is illustrated using four example problems involving small strain, large strain, self-weight and non-linear constitutive relationships. For these examples, estimates of total settlement, rate of settlement and void ratio distribution obtained using CS2 are comparable to other available analytical and numerical solutions. The piecewise-linear methodology is thus a viable alternative to more traditional modelling approaches for large strain consolidation based on Lagrangian and material co-ordinates.
(3) Correction factors are presented for the conventional Terzaghi theory which account for the effect of vertical strain of the compressible layer during the consolidation process. For any given average degree of consolidation, U_{avg}, the ratio of elapsed time for finite strain and infinitesimal strain solutions, t^r, is a single-valued function of the final strain, S^*, at the end of consolidation. As S^* increases, the drainage path length decreases during consolidation for the finite strain case, which consequently decreases t^r. For $S^* = 0{\cdot}4$, the error introduced by the infinitesimal strain approximation is approximately 30 per cent. The ratio of maximum excess pore pressures for finite strain and infinitesimal strain solutions, u^r_{ex}, increases with S^*. Thus, for any given U_{avg}, the maximum excess pore pressure in the case of finite-strain consolidation is larger than the corresponding maximum excess pore pressure predicted using conventional theory. For $S^* = 0{\cdot}4$, the error introduced by the infinitesimal strain approximation for maximum excess pore pressure is less than 5 per cent.

This paper has presented fundamental concepts involved in the development of CS2. The comparatively simple formulation of CS2 makes it particularly advantageous for the incorporation of additional modifications, including layered systems, time-dependent applied loads and piezometric groundwater levels, depth-dependent and pre-existing initial excess pore pressures, log-linear constitutive relationships including a preconsolidation pressure and unloading/reloading effects. Work is currently in progress to incorporate these features into the next version of the code, CS3. The subject of this research is left to a future paper.

ACKNOWLEDGEMENTS

The authors extend their sincere appreciation to Dr. Joseph P. Morris for his assistance in the development of equation (31) as an alternative time-increment criterion for CS2. In addition, the comments of Dr. G. A. Leonards regarding the formulation of the model were most helpful. This

research was supported by Grants No. CMS-9309566 and No. CMS-9622644 from the U. S. National Science Foundation. This support is gratefully acknowledged.

APPENDIX I

Notation

a_v	coefficient of compressibility
c_F	finite strain coefficient of consolidation
c_v	coefficient of consolidation
e	void ratio
G_s	specific gravity of solids
h	total head
H	height of compressible layer
H_w	elevation of groundwater table
i	hydraulic gradient
j	element co-ordinate
k	hydraulic conductivity
k_s	series hydraulic conductivity
L	height of element
n	porosity
q_o	initial effective overburden stress at top of compressible layer
R_j	number of elements
R_m	number of data points for compressibility curve
R_n	number of data points for hydraulic conductivity curve
S	settlement
t	time
u	pore pressure
u_{ex}	excess pore pressure
U_{avg}	average degree of consolidation
U_e	local degree of consolidation based on void ratio
U_u	local degree of consolidation based on excess pore pressure
$\tilde{v}_f$	absolute seepage velocity of fluid relative to datum
v_{rf}	discharge velocity of fluid relative to solid phase
$\tilde{v}_{rf}$	seepage velocity of fluid relative to solid phase
$\tilde{v}_s$	absolute velocity of solids relative to datum
W_s	weight of solids
z	vertical co-ordinate
α	constant
Δq	change in effective overburden stress at top of compressible layer
Δt	time increment
Δz	vertical node spacing
$\Delta \tau$	dimensionless time increment
γ	saturated unit weight
γ_w	unit weight of water
σ	vertical total stress

σ'	vertical effective stress
τ	dimensionless time

Superscripts

r	ratio of elapsed time or maximum excess pore pressure
t	time
$\Delta\tau$	dimensionless time increment
τ	dimensionless time
*	dimensionless quantity
^	input data for compressibility curve
‾	input data for hydraulic conductivity curve

Subscripts

b	bottom of compressible layer
f	final value
final	termination criterion
i	summation index
j	jth element
m	mth data point for compressibility curve
max	maximum value
n	nth data point for hydraulic conductivity curve
o	initial value
q_o	value at $\sigma' = q_o$
R_j	element $j = R_j$
1	element $j = 1$
90	value at $U_{avg} = 90$ per cent

APPENDIX II

CS2 source code (not including output statements)

```
program CS2;        {Piecewise-linear finite difference model for 1-D large strain consolidation}
                    {Patrick J. Fox and James D. Berles, Purdue University, 15 October 1996}
                    {All quantities in this code are dimensionless}
 const
  alpha = 0.4;                          {constant for time increment calculation}
  maxdatapts = 50;       {maximum number of data points for each constitutive relation}
  maxel = 201;                                     {maximum number of elements}
 var
  inn: text;                                                   {input file}
  topdrained, bottomdrained: boolean;               {boundary drainage conditions}
  j, m, n: integer;                                {loop indices for procedures}
  Rj, Rm, Rn: integer;   {number of: elements, compr. data pts., hydr. cond. data pts.}
  H, Hw, S, St: double;   {layer ht., groundwater elev., total settlement, settlement}
  Lo, Gs: double;                    {initial element ht., specific gravity of solids}
  t, dt, qo, dq: double;      {time, time increment, initial stress, stress increment}
```

```
  ib, vrfb: double;                    {hydraulic gradient and relative discharge vel. at bottom of layer}
  Uavg, tfinal, Uavgfinal: double;              {average degree of consolidation, termination criteria}
  eh, Esh: array[1..maxdatapts] of double;      {void ratio and effective stress pts. for compr. curve}
  eb, kb: array[1..maxdatapts] of double;        {void ratio and hydr. cond. pts. for hydr. cond. curve}
  L, eo, ef, e: array[1..maxel] of double;  {element ht., initial void ratio, final void ratio, void ratio}
  z, Ts, Es: array[1..maxel] of double;                {node elevation, total stress, effective stress}
  k, ks: array[1..maxel] of double;              {hydraulic conductivity, series hydraulic conductivity}
  u, ht, i, vrf: array[1..maxel] of double;{pore press., total head, hydr. gradient, rel. discharge vel.}
  av, gsat: array[1..maxel] of double;                {coeff. of compressibility, saturated unit weight}
  Ue, Uu: array[1..maxel] of double;       {local degree of consol. based on void ratio and pore press.}

procedure Input;
begin
  readln(inn, Rj, Rm, Rn);
  for m := 1 to Rm do readln(inn, eh[m], Esh[m]);             {compressibility curve:  high e to low e}
  for n := 1 to Rn do readln(inn, eb[n], kb[n]);         {hydraulic conductivity curve:  high e to low e}
  readln(inn, qo, dq, Gs, Hw, topdrained, bottomdrained,  tfinal, Uavgfinal)
end;

procedure Initial_Calculations;
 var
  newEs: double;
begin
  H := 1;
  Lo := 1 / Rj;
  for j := 1 to Rj do L[j] := Lo;                             {all elements have same initial thickness}
  if qo < Esh[1] then writeln('Compressibility curve does not cover required stress range');
  newEs := qo;
  for j := 1 to Rj do                                                               {initial void ratios}
   repeat
    Es[j] := newEs;
    if Es[j] > Esh[Rm] then writeln('Compressibility curve does not cover required stress range');
    m := 1;
    repeat m := m + 1
    until Esh[m] >= Es[j];
    eo[j] := eh[m - 1] + (eh[m] - eh[m - 1]) / (Esh[m] - Esh[m - 1]) * (Es[j] - Esh[m - 1]);
    gsat[j] :=  (Gs + eo[j]) / (1 + eo[j]);
    if j = 1 then newEs := qo + (gsat[1] - 1) * L[1] / 2
    else newEs := Es[j - 1] + (gsat[j - 1] - 1) * L[j - 1] / 2 + (gsat[j] - 1) * L[j] / 2

   until abs(newEs - Es[j]) < 1e-12;
  for j := 1 to Rj do e[j] := eo[j];
  for j := 1 to Rj do begin                                                           {final void ratios}
   if Es[j] + dq > Esh[Rm] then writeln('Compressibility curve does not cover required stress range');
   m := 1;
   repeat m := m + 1
   until Esh[m] >= Es[j] + dq;
   ef[j] := eh[m - 1] + (eh[m] - eh[m - 1]) / (Esh[m] - Esh[m - 1]) * (Es[j] + dq - Esh[m - 1])
   end;
  if (eo[1] > eb[1]) or (ef[Rj] < eb[Rn]) then
   writeln('Hydraulic conductivity curve does not cover required void ratio range');
  S := 0;
  for j := 1 to Rj do S := S + Lo * (eo[j] - ef[j]) / (1 + eo[j])                      {total settlement}
end;
```



```
procedure Node_Elevations;                              {fixed Eulerian coordinate system}
begin
 z[Rj] := 0.5 * L[Rj];
 for j := (Rj - 1) downto 1 do z[j] := z[j + 1] + 0.5 * L[j + 1] + 0.5 * L[j]
end;

procedure Stresses;
begin
 for j := 1 to Rj do  begin
  gsat[j] := (Gs + e[j]) / (1 + e[j]);
  if j = 1 then Ts[1] := Hw - H + qo + dq  + gsat[1] * L[1] / 2                 {total stress}
  else Ts[j] := Ts[j - 1] + gsat[j - 1] * L[j - 1] / 2 + gsat[j] * L[j] / 2;
  m := 1;
  repeat m := m + 1
  until e[j] >= eh[m];
  av[j] := -(eh[m] - eh[m - 1]) / (Esh[m] - Esh[m - 1]);       {coefficient of compressibility}
  Es[j] := Esh[m - 1] + (eh[m - 1] - e[j]) / av[j]                            {effective stress}
  end
end;

procedure Hydraulic_Conductivity;
begin
 for j := 1 to Rj do  begin      {hydraulic conductivity and series hydraulic conductivity}
  n := 1;
  repeat n := n + 1
  until e[j] >= eb[n];
  k[j] := kb[n - 1] + (kb[n] - kb[n - 1]) / (eb[n] - eb[n - 1]) * (e[j] - eb[n - 1])
  end;
 for j := 2 to Rj do ks[j] := (L[j - 1] + L[j]) * k[j - 1] * k[j] / (L[j - 1] * k[j] + L[j] * k[j - 1])
end;

procedure Flow;
begin
 for j := 1 to Rj do  begin
  u[j] := Ts[j] - Es[j];                                                     {pore pressure}
  ht[j] := z[j] + u[j]                                                          {total head}
  end;
 if topdrained then i[1] := (Hw - ht[1]) / (H - z[1])                   {hydraulic gradient}
 else i[1] := 0;
 for j := 2 to Rj do i[j] := (ht[j - 1] - ht[j]) / (z[j - 1] - z[j]);
 if bottomdrained then ib := (ht[Rj] - Hw) / z[Rj]
 else ib := 0;
 vrf[1] := -k[1] * i[1];                                        {relative discharge velocity}
 for j := 2 to Rj do vrf[j] := -ks[j] * i[j];
 vrfb := -k[Rj] * ib
end;

procedure Time_Increment;
begin
 dt := alpha * sqr(L[1]) * av[1] / k[1] / (1 + e[1]);
 for j := 2 to Rj do                                               {minimum time increment}
  if dt > alpha * sqr(L[j]) * av[j] / k[j] / (1 + e[j]) then
   dt := alpha * sqr(L[j]) * av[j] / k[j] / (1 + e[j]);
```

```
 for j := 1 to Rj - 1 do
  if dt > abs(0.01 * Lo * (eo[j] - ef[j]) / (1 + eo[j]) / (vrf[j] - vrf[j + 1])) then
   dt := abs(0.01 * Lo * (eo[j] - ef[j]) / (1 + eo[j]) / (vrf[j] - vrf[j + 1]));
 if dt > abs(0.01 * Lo * (eo[Rj] - ef[Rj]) / (1 + eo[Rj]) / (vrf[Rj] - vrfb)) then
   dt := abs(0.01 * Lo * (eo[Rj] - ef[Rj]) / (1 + eo[Rj]) / (vrf[Rj] - vrfb))
end;

procedure Settlement;
begin
 H := 0;
 for j := 1 to Rj - 1 do begin
  L[j] := L[j] - (vrf[j] - vrf[j + 1]) * dt;                 {new element height}
  H := H + L[j];                                              {new layer height}
  e[j] := L[j] / Lo * (1 + eo[j]) - 1                         {new void ratio}
  end;
 L[Rj] := L[Rj] - (vrf[Rj] - vrfb) * dt;
 H := H + L[Rj];
 e[Rj] := L[Rj] / Lo * (1 + eo[Rj]) - 1;
 St := 1 - H                                                  {new settlement}
end;

procedure Degree_of_Consolidation;
begin
 for j := 1 to Rj do begin
  Uu[j] := 100 * (1 - (u[j] - (Hw - z[j]) ) / dq);   {local degree of consolidation based on uex, %}
  Ue[j] := 100 * (eo[j] - e[j]) / (eo[j] - ef[j])     {local degree of consolidation based on e, %}
  end;
 Uavg := 100 * (St / S)                                {average degree of consolidation, %}
end;

begin                                                         {main program}
 reset(inn, 'CS2 Input File');
 Input;
 Initial_Calculations;
 t := 0;
 repeat
  Node_Elevations;
  Stresses;
  Hydraulic_Conductivity;
  Flow;
  Time_Increment;
  t := t + dt;
  Settlement;
  Degree_of_Consolidation
 until (t >= tfinal) or (Uavg >= Uavgfinal)                   {terminate?}
end.
```

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